

Omni-Lie Superalgebras and Lie 2-superalgebras

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Abstract We introduce the notion of omni-Lie superalgebra as a super version of the omni-Lie algebra introduced by Weinstein. This algebraic structure gives a nontrivial example of Leibniz superalgebra and Lie 2-superalgebra. We prove that there is a one-to-one correspondence between Dirac structures of the omni-Lie superalgebra and Lie superalgebra structures on subspaces of a super vector space.

1 Introduction

In [17], Weinstein introduced the notion of omni-Lie algebra, which can be regarded as the linearization of the Courant bracket. An omni-Lie algebra associated to a vector space V is the direct sum space $\mathfrak{gl}(V) \oplus V$ together with the nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle$ and the skew-symmetric bracket $[\cdot, \cdot]$ given by

$$\langle A + u, B + v \rangle = \frac{1}{2}(Av + Bu),$$

and

$$[A + u, B + v] = [A, B] + \frac{1}{2}(Av - Bu).$$

The bracket $[\cdot, \cdot]$ does not satisfy the Jacobi identity so that an omni-Lie algebra is not a Lie algebra. An omni-Lie algebra is actually a Lie 2-algebra since Roytenberg and Weinstein proved that every Courant algebroid gives rise to a Lie 2-algebra ([13]). Recently, omni-Lie algebras are studied from several aspects ([4], [9], [16]) and are generalized to omni-Lie algebroids and omni-Lie 2-algebras in [5, 6, 15]. The corresponding Dirac structures are also studied therein.

In this paper, we introduce the notion of omni-Lie superalgebra, which is the super analogue of omni-Lie algebra. We also study Dirac structures of omni-Lie superalgebra in order to characterize all possible Lie superalgebra structures on a super vector space. We prove that omni-Lie superalgebra is a Leibniz superalgebra as well as a Lie 2-superalgebra, which is a super version of Lie 2-algebra or a 2-term L_∞ -algebra ([2, 8, 10]).

The paper is organized as follows. In Section 2, we recall some basic facts for Lie superalgebras. In Section 3, we define omni-Lie superalgebra on $\mathcal{E} = \mathfrak{gl}(V) \oplus V$ for a super vector space V and study Dirac structures. In Section 4, we prove an omni-Lie superalgebra is a Lie 2-superalgebra.

2 Lie Superalgebras and Leibniz Superalgebras

We first recall some facts and definitions about Lie superalgebras, basic reference is Kac [7]. We work on a fixed field \mathbb{K} of characteristic 0.

A super vector space V is a \mathbb{Z}_2 -graded vector space with a direct sum decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$. An element $x \in V_{\bar{0}} \cup V_{\bar{1}}$ is called homogeneous. The degree of a homogeneous element $x \in V_{\alpha}$, $\alpha \in \mathbb{Z}_2$ is defined by $|x| = \alpha$. A morphism between two super vector spaces, V and W , is a grade-preserving linear map:

$$f : V \longrightarrow W, \quad f(V_{\alpha}) \subseteq W_{\alpha}, \quad \forall \alpha \in \mathbb{Z}_2.$$

The direct sum $V \oplus W$ is graded by

$$(V \oplus W)_{\bar{0}} = V_{\bar{0}} \oplus W_{\bar{0}}, \quad (V \oplus W)_{\bar{1}} = V_{\bar{1}} \oplus W_{\bar{1}},$$

and the tensor product $V \otimes W$ is graded by

$$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}), \quad (V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}}).$$

Definition 2.1. A Lie superalgebra is a super vector space (i.e. \mathbb{Z}_2 -graded vector space) $L = L_{\bar{0}} \oplus L_{\bar{1}}$ together with a bracket $[\cdot, \cdot] : L \otimes L \rightarrow L$ satisfies,

(i) *graded condition:* $[L_{\alpha}, L_{\beta}] \subseteq L_{\alpha+\beta} \quad \forall \alpha, \beta \in \mathbb{Z}_2$,

(ii) *super skew-symmetry:*

$$[x, y] + (-1)^{|x||y|}[y, x] = 0, \quad (1)$$

(iii) *super Jacobi identity:*

$$J_1 := (-1)^{|z||x|}[[x, y], z] + (-1)^{|x||y|}[[y, z], x] + (-1)^{|y||z|}[[z, x], y] = 0, \quad (2)$$

where $x, y, z \in L$ are homogeneous elements of degree $|x|, |y|, |z|$ respectively.

One can rewrite the super Jacobi identity in another form:

$$J_2 := -(-1)^{|z||x|}J_1 = [x, [y, z]] - [[x, y], z] - (-1)^{|x||y|}[y, [x, z]] = 0, \quad (3)$$

which will be convenient for our use below.

Example 2.2. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative superalgebra with multiplication $A_{\alpha}A_{\beta} \subseteq A_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$. Define the bracket:

$$[x, y] := xy - (-1)^{|x||y|}yx, \quad \forall x, y \in A. \quad (4)$$

Then $(A, [\cdot, \cdot])$ is a Lie superalgebra which is denoted by A_L .

Example 2.3. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space, then we have the general linear Lie superalgebra

$$\mathfrak{gl}(V) = \mathfrak{gl}(V)_{\bar{0}} \oplus \mathfrak{gl}(V)_{\bar{1}},$$

such that

$$\begin{aligned}\mathfrak{gl}(V)_{\bar{0}} &= \text{Hom}(V_{\bar{0}}, V_{\bar{0}}) \oplus \text{Hom}(V_{\bar{1}}, V_{\bar{1}}), \\ \mathfrak{gl}(V)_{\bar{1}} &= \text{Hom}(V_{\bar{0}}, V_{\bar{1}}) \oplus \text{Hom}(V_{\bar{1}}, V_{\bar{0}}),\end{aligned}$$

and the bracket is given by (4). When $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = n$, $\mathfrak{gl}(V)$ is usually denoted by $\mathfrak{gl}(m|n)$.

A homomorphism between two Lie superalgebras $(L, [\cdot, \cdot])$ and $(L', [\cdot, \cdot]')$ is linear map $\varphi : L \rightarrow L'$ such that

$$\varphi(L_{\alpha}) \subseteq L'_{\alpha}, \quad \varphi([x, y]) = [\varphi(x), \varphi(y)]', \quad \forall x, y \in L, \quad \forall \alpha \in \mathbb{Z}_2.$$

A super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is called a module of a Lie superalgebra L or, equivalently, say that L acts on V if there is a homomorphism $\rho : L \rightarrow \mathfrak{gl}(V)$, i.e.,

$$\rho([x, y])v = \rho(x)\rho(y)v - (-1)^{|x||y|}\rho(y)\rho(x)v. \quad (5)$$

For simplicity, one often writes $xv = \rho(x)v$ to denote such an action. A new Lie superalgebra can be constructed as follows.

Proposition 2.4. [14] Let L be a Lie superalgebra with an action on V . Define a bracket on $L \oplus V$ by

$$[x + u, y + v] := [x, y]_L + xv - (-1)^{|u||y|}yu. \quad (6)$$

Then $(L \oplus V, [\cdot, \cdot])$ becomes a Lie superalgebra, denoted by $L \ltimes V$ and called semidirect product of L and V .

In [12], Loday introduced a new algebraic structure, which is usually called Leibniz algebra. Its super version is as follows.

Definition 2.5. [1] A Leibniz superalgebra is a super vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ together with a morphism $\circ : L \otimes L \rightarrow L$ satisfying $L_{\alpha} \circ L_{\beta} \subseteq L_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$, and the super Leibniz rule:

$$x \circ (y \circ z) = (x \circ y) \circ z + (-1)^{|x||y|}y \circ (x \circ z), \quad (7)$$

for all homogeneous elements $x, y, z \in L$.

By definition, it is easy to see that a Leibniz superalgebra is just a Lie superalgebra if the operation " \circ " is super skew-symmetric. In this case, the super Leibniz rule above is actually the super Jacobi identity for J_2 given by (3).

3 Omni-Lie Superalgebras and Dirac Structures

Let V be a super vector space, recall that the space $\mathfrak{gl}(V) \oplus V$ has a \mathbb{Z}_2 -grading

$$\mathcal{E} := \mathfrak{gl}(V) \oplus V = (\mathfrak{gl}(V)_0 \oplus V_0) \oplus (\mathfrak{gl}(V)_1 \oplus V_1).$$

Like in [9], we define an operation \circ on $\mathfrak{gl}(V) \oplus V$ as follows:

$$(A + x) \circ (B + y) = [A, B] + Ay. \quad (8)$$

Then we have

Proposition 3.1. *(\mathcal{E}, \circ) is a Leibniz superalgebra.*

Proof. To check the super Leibniz rule (7) holds on \mathcal{E} under the operation \circ , let $e_1 = A + x, e_2 = B + y, e_3 = C + z$ be homogenous elements of degree $|A| = |x|, |B| = |y|$ and $|C| = |z|$. By definition, we have

$$\begin{aligned} & \{e_1 \circ e_2\} \circ e_3 - e_1 \circ \{e_2 \circ e_3\} - (-1)^{|x||y|} e_2 \circ \{e_1 \circ e_3\} \\ &= ([A, B] + Ay) \circ (C + z) - (A + x) \circ ([B, C] + Bz) \\ & \quad - (-1)^{|x||y|} (B + y) \circ ([A, C] + Az) \\ &= [[A, B], C] - [A, [B, C]] - (-1)^{|x||y|} [B, [A, C]] \\ & \quad + [A, B]z - ABz - (-1)^{|x||y|} BAz \\ &= 0, \end{aligned}$$

where the equality holds because $\mathfrak{gl}(V)$ is a Lie superalgebra acting on V . \square

Note that the above operation is not super skew-symmetric, we can define a super skew-symmetric bracket on $\mathcal{E} = \mathfrak{gl}(V) \oplus V$ as its skew symmetrization:

$$\begin{aligned} \llbracket A + x, B + y \rrbracket &\triangleq \frac{1}{2}((A + x) \circ (B + y) - (B + y) \circ (A + x)) \\ &= [A, B] + \frac{1}{2}(Ay - (-1)^{|x||y|} Bx), \end{aligned} \quad (9)$$

and define a V -valued inner product, i.e., a non-degenerated super symmetric bilinear form:

$$\langle A + x, B + y \rangle \triangleq \frac{1}{2}(Ay + (-1)^{|x||y|} Bx). \quad (10)$$

We call the triple $(\mathcal{E}, \llbracket \cdot, \cdot \rrbracket, \langle \cdot, \cdot \rangle)$ an **omni-Lie superalgebra**. Without the factor $1/2$ in bracket $\llbracket \cdot, \cdot \rrbracket$, this would be the semidirect product Lie superalgebra for the action of $\mathfrak{gl}(V)$ on V described in Proposition 2.4. With the factor $1/2$, the bracket does not satisfy the super Jacobi identity, which leads to the concept of Lie 2-superalgebra defined in the next section. Next we compute the Jacobiator for this bracket.

Proposition 3.2. *For $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \mathcal{E}$, define*

$$\begin{aligned} T(e_1, e_2, e_3) &:= \frac{1}{3} \{ (-1)^{|z||x|} \langle \llbracket e_1, e_2 \rrbracket, e_3 \rangle + (-1)^{|x||y|} \langle \llbracket e_2, e_3 \rrbracket, e_1 \rangle \\ & \quad + (-1)^{|y||z|} \langle \llbracket e_3, e_1 \rrbracket, e_2 \rangle \}. \end{aligned}$$

Let J_1 denote the Jacobiator given in (2) for the bracket $\llbracket \cdot, \cdot \rrbracket$ on \mathcal{E} , then we have

$$J_1(e_1, e_2, e_3) = T(e_1, e_2, e_3).$$

Proof. We compute both the sides as follows:

$$\begin{aligned}
& J_1(e_1, e_2, e_3) \\
&= (-1)^{|z||x|} \llbracket \llbracket A+x, B+y \rrbracket, C+z \rrbracket + \text{c.p.} \\
&= \llbracket (-1)^{|z||x|} [A, B] + \frac{1}{2} (-1)^{|z||x|} (Ay - (-1)^{|x||y|} Bx), C+z \rrbracket + \text{c.p.} \\
&= (-1)^{|z||x|} \llbracket [A, B], C \rrbracket + \text{c.p.} \\
&\quad + \frac{1}{2} \left((-1)^{|z||x|} [A, B]z - \frac{1}{2} (-1)^{|z||x|} (-1)^{(|x|+|y|)|z|} C (Ay - (-1)^{|x||y|} Bx) \right) \\
&\quad + \frac{1}{2} \left((-1)^{|x||y|} [B, C]x - \frac{1}{2} (-1)^{|x||y|} (-1)^{(|y|+|z|)|x|} A (Bz - (-1)^{|y||z|} Cy) \right) \\
&\quad + \frac{1}{2} \left((-1)^{|y||z|} [C, A]y - \frac{1}{2} (-1)^{|z||y|} (-1)^{(|z|+|x|)|y|} B (Cx - (-1)^{|z||x|} Az) \right) \\
&= \frac{1}{4} (-1)^{|z||x|} ABz - \frac{1}{4} (-1)^{|z||x|} (-1)^{|x||y|} BAz + \frac{1}{4} (-1)^{|y||z|} CAy \\
&\quad - \frac{1}{4} (-1)^{|y||z|} (-1)^{|x||y|} CBx + \frac{1}{4} (-1)^{|x||y|} BCx - \frac{1}{4} (-1)^{|z||x|} (-1)^{|y||z|} ACy, \\
& \\
& T(e_1, e_2, e_3) \\
&= \frac{1}{3} (-1)^{|z||x|} \langle \llbracket A+x, B+y \rrbracket, C+z \rangle + \text{c.p.} \\
&= \frac{1}{3} (-1)^{|z||x|} \langle [A, B] + \frac{1}{2} (Ay - (-1)^{|x||y|} Bx), C+z \rangle + \text{c.p.} \\
&= \frac{1}{6} (-1)^{|z||x|} ([A, B]z + \frac{1}{2} (-1)^{(|x|+|y|)|z|} C (Ay - (-1)^{|x||y|} Bx)) + \text{c.p.} \\
&= \frac{1}{6} (-1)^{|z||x|} ABz - \frac{1}{6} (-1)^{|z||x|} (-1)^{|x||y|} BAz + \frac{1}{12} (-1)^{|y||z|} CAy \\
&\quad - \frac{1}{12} (-1)^{|y||z|} (-1)^{|x||y|} CBx + \frac{1}{6} (-1)^{|x||y|} BCx - \frac{1}{6} (-1)^{|x||y|} (-1)^{|y||z|} CBx \\
&\quad + \frac{1}{12} (-1)^{|z||x|} ABz - \frac{1}{12} (-1)^{|z||x|} (-1)^{|y||z|} ACy + \frac{1}{6} (-1)^{|y||z|} CAy \\
&\quad - \frac{1}{6} (-1)^{|y||z|} (-1)^{|z||x|} ACy + \frac{1}{12} (-1)^{|x||y|} BCx - \frac{1}{12} (-1)^{|x||y|} (-1)^{|z||x|} BAz \\
&= \frac{1}{4} (-1)^{|z||x|} ABz - \frac{1}{4} (-1)^{|z||x|} (-1)^{|x||y|} BAz + \frac{1}{4} (-1)^{|y||z|} CAy \\
&\quad - \frac{1}{4} (-1)^{|y||z|} (-1)^{|x||y|} CBx + \frac{1}{4} (-1)^{|x||y|} BCx - \frac{1}{4} (-1)^{|z||x|} (-1)^{|y||z|} ACy.
\end{aligned}$$

Thus, the two sides are equal. \square

The bracket $\llbracket \cdot, \cdot \rrbracket$ does not satisfy the super Jacobi identity so that an omni-Lie superalgebra is not a Lie superalgebra. However, all possible Lie superalgebra structures on V can be characterized by means of the omni-Lie superalgebra.

For a bilinear operation ω on V such that $\omega : V_\alpha \times V_\beta \rightarrow V_{\alpha+\beta}$, we define the adjoint operator

$$\text{ad}_\omega : V_\alpha \rightarrow \mathfrak{gl}(V)_\alpha, \quad \text{ad}_\omega(x)(y) = \omega(x, y) \in V_{\alpha+\beta}$$

where $x \in V_\alpha, y \in V_\beta$. Then the graph of the adjoint operator:

$$\mathcal{F}_\omega = \{\text{ad}_\omega x + x ; \forall x \in V\} \subset \mathcal{E} = \mathfrak{gl}(V) \oplus V$$

is a super subspace of \mathcal{E} . Denote \mathcal{F}_ω^\perp the orthogonal complement of \mathcal{F}_ω in \mathcal{E} with respect to the super symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathcal{E} given in (10).

Proposition 3.3. *With the above notations, (V, ω) is a Lie superalgebra if and only if its graph \mathcal{F}_ω is maximal isotropic, i.e. $\mathcal{F}_\omega = \mathcal{F}_\omega^\perp$, and is closed with respect to the bracket $\llbracket \cdot, \cdot \rrbracket$.*

Proof. First we see that

$$\begin{aligned} \langle \text{ad}_\omega(x) + x, \text{ad}_\omega(y) + y \rangle &= \frac{1}{2}(\text{ad}_\omega(x)y + (-1)^{|x||y|} \text{ad}_\omega(y)x) \\ &= \frac{1}{2}(\omega(x, y) + (-1)^{|x||y|} \omega(y, x)). \end{aligned}$$

This means that ω is super skew-symmetric if and only if its graph is isotropic, i.e. $\mathcal{F}_\omega \subseteq \mathcal{F}_\omega^\perp$. Moreover, by dimension analysis, we have \mathcal{F}_ω is maximal isotropic.

Next let $[x, y] := \omega(x, y)$, we shall check that the super Jacobi identity on V is satisfied if and only if \mathcal{F}_ω is closed under bracket (9) on \mathcal{E} . In fact,

$$\begin{aligned} \llbracket \text{ad}_\omega(x) + x, \text{ad}_\omega(y) + y \rrbracket &= [\text{ad}_\omega(x), \text{ad}_\omega(y)] + \frac{1}{2}(\text{ad}_\omega(x)y - (-1)^{|x||y|} \text{ad}_\omega(y)x) \\ &= [\text{ad}_\omega(x), \text{ad}_\omega(y)] + \frac{1}{2}(\omega(x, y) - (-1)^{|x||y|} \omega(y, x)) \\ &= [\text{ad}_\omega(x), \text{ad}_\omega(y)] + \omega(x, y). \end{aligned}$$

Thus this bracket is closed if and only if

$$[\text{ad}_\omega(x), \text{ad}_\omega(y)] = \text{ad}_\omega(\omega(x, y)).$$

In this case, for $\forall z \in V$, we have

$$\begin{aligned} &[\text{ad}_\omega(x), \text{ad}_\omega(y)](z) - \text{ad}_\omega(\omega(x, y))(z) \\ &= \text{ad}_\omega(x) \text{ad}_\omega(y)(z) - (-1)^{|x||y|} \text{ad}_\omega(y) \text{ad}_\omega(x)(z) - \text{ad}_\omega(\omega(x, y))(z) \\ &= \text{ad}_\omega(x) \omega(y, z) - (-1)^{|x||y|} \text{ad}_\omega(y) \omega(x, z) - \omega(\omega(x, y), z) \\ &= \omega(x, \omega(y, z)) - (-1)^{|x||y|} \omega(y, \omega(x, z)) - \omega(\omega(x, y), z) \\ &= [x, [y, z]] - (-1)^{|x||y|} [y, [x, z]] - [[x, y], z] \\ &= 0. \end{aligned}$$

This is exactly the super Jacobi identity on V . Therefore, the conclusion follows from Definition 2.1. \square

In [3], quadratic Lie superalgebras are studied for a given inner product B on V . In this case, one has the orthogonal Lie superalgebra $\mathfrak{o}(V) \subset \mathfrak{gl}(V)$ and it is easy to see that (V, ω, B) is a quadratic Lie superalgebra if and only if ω satisfies the two conditions in Proposition 3.3 above as well as $\text{ad}_\omega x \in \mathfrak{o}(V), \forall x \in V$.

Definition 3.4. *A Dirac structure L of the omni-Lie superalgebra $(\mathfrak{gl}(V) \oplus V, \llbracket \cdot, \cdot \rrbracket, \langle \cdot, \cdot \rangle)$ is a maximal isotropic subspace ($L = L^\perp$) and closed under the bracket $\llbracket \cdot, \cdot \rrbracket$.*

Remark 3.5. *According to Proposition 3.2, for a Dirac structure L , we have*

$$J_1(e_1, e_2, e_3) = T(e_1, e_2, e_3) = 0, \quad \forall e_i \in L.$$

Thus a Dirac structure is a Lie superalgebra, though omni-Lie superalgebra is not for itself. In fact, a Dirac structure is also a Leibniz subalgebra under the operation \circ .

By Proposition 3.3, (V, ω) is a Lie superalgebra if and only if \mathcal{F}_ω is a Dirac structure of the omni-Lie superalgebra $\mathfrak{gl}(V) \oplus V$. In order to give a general characterization for all Dirac structures of \mathcal{E} , we adapt the theory of characteristic pairs developed in [11] (see also [15]).

For a maximal isotropic subspace $L \subset \mathfrak{gl}(V) \oplus V$, set the subspace $D = L \cap \mathfrak{gl}(V)$. Define $D^0 \subset V$ to be the null space of D :

$$D^0 = \{x \in V \mid X(x) = 0, \forall X \in D\}.$$

It is easy to see that $D = (D^0)^0$.

Lemma 3.6. *With notations above, a subspace L is maximal isotropic if and only if L is of the form*

$$L = D \oplus \mathcal{F}_{\pi|_{D^0}} = \{X + \pi(x) + x \mid X \in D, x \in D^0\}, \quad (11)$$

where $\pi : V \rightarrow \mathfrak{gl}(V)$ is a super skew-symmetric map.

Proof. In the following, we also denote $\pi(x, y) = \pi(x)(y) \in V$ for convenience. First suppose that L is given by (11), then

$$\begin{aligned} & \langle X + \pi(x) + x, Y + \pi(y) + y \rangle \\ &= \frac{1}{2} \{X(y) + \pi(x, y) + (-1)^{|x||y|} Y(x) + (-1)^{|x||y|} \pi(y, x)\} \\ &= \frac{1}{2} \{\pi(x, y) + (-1)^{|x||y|} \pi(y, x)\} \\ &= 0, \quad \forall X + \pi(x) + x, Y + \pi(y) + y \in L, \end{aligned}$$

since $\pi : V \rightarrow \mathfrak{gl}(V)$ is super skew-symmetric so that L is isotropic. Next we prove that L is maximal isotropic. For $\forall Z + z \in L^\perp$,

$$\langle X, Z + z \rangle = X(z) = 0, \quad \forall X \in D \Rightarrow z \in D^0.$$

Moreover, $\forall x \in D^0$, the equality below

$$\begin{aligned} \langle X + \pi(x) + x, C + z \rangle &= X(z) + \pi(x)(z) + (-1)^{|x||z|} Cx \\ &= (-1)^{|x||z|} (C - \pi(x)(z))(x) = 0, \end{aligned}$$

implies that $C - \pi(z) \triangleq Z \in D$. Thus

$$C + z = Z + \pi(z) + z \in L = D \oplus \mathcal{F}_{\pi|_{D^0}} \Rightarrow L = L^\perp.$$

The converse part is straightforward so we omit the details. \square

The proof of the following Lemma is skipped since it is straightforward and similar to that in [15].

Lemma 3.7. *Let (D, π) be given above. Then L is a Dirac structure if and only if the following conditions are satisfied:*

- (1) D is a subalgebra of $\mathfrak{gl}(V)$;
- (2) $\pi(\pi(x, y)) - [\pi(x), \pi(y)] \in D, \quad \forall x, y \in D^0$;
- (3) $\pi(x, y) \in D^0, \quad \forall x, y \in D^0$.

Such a pair (D, π) is called a **characteristic pair** of a Dirac structure L . By means of the two lemmas above, we can mention the main result in this section.

Theorem 3.8. *There is a one-to-one correspondence between Dirac structures of the omni-Lie superalgebra $(\mathfrak{gl}(V) \oplus V, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ and Lie superalgebra structures on subspaces of V .*

Proof. For any Dirac structure $L = D \oplus \mathcal{F}_{\pi|_{D^0}}$, a Lie superalgebra structure on D^0 is as follows:

$$[x, y]_{D^0} \triangleq \pi(x, y) \in D^0, \quad \forall x, y \in D^0.$$

It easy to see that this is a super skew-symmetric map. For super Jacobi identity, we have for all $x, y, z \in D^0$,

$$\begin{aligned} [[x, y]_{D^0}, z]_{D^0} &= \pi([x, y]_{D^0})(z) = \pi((\pi(x)(y))(z)) = [\pi(x), \pi(y)](z) \\ &= \pi(x)(\pi(y)(z)) - (-1)^{|x||y|} \pi(y)(\pi(x)(z)) \\ &= [x, [y, z]_{D^0}]_{D^0} - (-1)^{|x||y|} [y, [x, z]_{D^0}]_{D^0}. \end{aligned}$$

Thus we get a Lie superalgebra $(D^0, [\cdot, \cdot]_{D^0})$.

Conversely, for any Lie superalgebra $(W, [\cdot, \cdot]_W)$ on a subspace W of V . Define D by

$$D = W^0 \triangleq \{X \in \mathfrak{gl}(V) \mid X(x) = 0, \quad \forall x \in W\}.$$

Then $D^0 = (W^0)^0 = W$. Since Lie superalgebra structure $[\cdot, \cdot]_W$ gives a super skew symmetric morphism:

$$\text{ad} : W \rightarrow \mathfrak{gl}(W), \quad \text{ad}_x(y) = [x, y]_W,$$

we take a super skew symmetric morphism $\pi : V \rightarrow \mathfrak{gl}(V)$, as an extension of ad from $W = D^0$ to V . Thus we get a maximal isotropic subspace $L = D \oplus \mathcal{F}_{\pi|_W}$ from the pair (D, π) as in Lemma 3.6.

We shall prove that L is a Dirac structure. Firstly, $\forall X, Y \in D$ and $x \in W$, we have

$$[X, Y](x) = XY(x) - (-1)^{|X||Y|} YX(x) = 0,$$

which implies that D is a subalgebra of $\mathfrak{gl}(V)$.

Next step is to prove that L is closed under the bracket $\llbracket \cdot, \cdot \rrbracket$. Remember that $\pi|_W = \text{ad}$ and $[\cdot, \cdot]_W$ satisfies the super Jacobi identity, we obtain

$$[\text{ad}_x, \text{ad}_y] = \text{ad}_{[x, y]_W} = \text{ad}_{\text{ad}_x y}, \quad \forall x, y \in W.$$

For any $X \in D$ and $x, y \in W$, we have

$$[X, \text{ad}_x](y) = X([x, y]_W) - (-1)^{|X||x|}[x, X(y)] = 0,$$

thus $[X, \text{ad}_x] \in D$. On the other hand, we have

$$\begin{aligned} & \llbracket X + \text{ad}_x + x, Y + \text{ad}_y + y \rrbracket \\ = & [X, Y] + [X, \text{ad}_y] + [\text{ad}_x, Y] + [\text{ad}_x, \text{ad}_y] + \frac{1}{2}(\text{ad}_x(y) - (-1)^{|x||y|} \text{ad}_y(x)) \\ = & [X, Y] + [X, \text{ad}_y] + [\text{ad}_x, Y] + \text{ad}_{[x, y]_W} + [x, y]_W \\ \in & D \oplus \mathcal{F}_{\pi|_W}, \end{aligned}$$

Thus, we conclude that L is a Dirac structure. Finally, it is easy to see that the Dirac structure L is independent of the choice of extension π . This completes the proof. \square

4 Lie 2-superalgebras

The concept of Lie n -superalgebras is introduced in [8]. In particular, the axiom of a Lie 2-superalgebra can be expressed explicitly as follows:

Definition 4.1. A Lie 2-superalgebra $\mathcal{V} = (\mathcal{V}^1 \xrightarrow{d} \mathcal{V}^0, l_2, l_3)$ consists of the following data:

- two super vector spaces \mathcal{V}^0 and \mathcal{V}^1 together with a morphism $d: \mathcal{V}^1 \rightarrow \mathcal{V}^0$;
- a morphism $l_2 = [\cdot, \cdot]: \mathcal{V}^i \otimes \mathcal{V}^j \rightarrow \mathcal{V}^{i+j}$;
- a morphism $l_3: \mathcal{V}^0 \otimes \mathcal{V}^0 \otimes \mathcal{V}^0 \rightarrow \mathcal{V}^1$;

such that, $\forall x, y, z, w \in \mathcal{V}^0; \forall h, k \in \mathcal{V}^1$,

- (a) $[x, y] + (-1)^{|x||y|}[y, x] = 0$;
- (b) $[x, h] + (-1)^{|x||h|}[h, x] = 0$;
- (c) $[h, k] = 0$;
- (d) $l_3(x, y, z)$ is totally super skew-symmetric;
- (e) $d([x, h]) = [x, dh]$;
- (f) $[dh, k] = [h, dk]$;
- (g) $d(l_3(x, y, z)) = -[[x, y], z] + [x, [y, z]] + (-1)^{|y||z|}[[x, z], y]$;

$$\begin{aligned}
(h) \quad l_3(x, y, dh) &= -[[x, y], h] + [x, [y, h]] + (-1)^{|y||h|}[[x, z], h]; \\
(i) \quad \delta l_3(x, y, z, w) &:= [x, l_3(y, z, w)] - (-1)^{|x||y|}[y, l_3(x, z, w)] \\
&\quad + (-1)^{(|x|+|y|)|z|}[z, l_3(x, y, w)] - [l_3(x, y, z), w] - l_3([x, y], z, w) \\
&\quad + (-1)^{|y||z|}l_3([x, z], y, w) - (-1)^{(|y|+|z|)|w|}l_3([x, w], y, z) \\
&\quad - l_3(x, [y, z], w) + (-1)^{|z||w|}l_3(x, [y, w], z) - l_3(x, y, [z, w]) = 0.
\end{aligned}$$

This is the super analogue of a 2-term L_∞ -algebra which is equivalent to a Lie 2-algebra (see [2] for more details). Here we use the terminology Lie 2-superalgebra instead of 2-term L_∞ -superalgebra.

Now, for a super vector space V , let

$$\mathcal{V}^0 = \mathfrak{gl}(V) \oplus V, \quad \mathcal{V}^1 = V, \quad d = i : V \hookrightarrow \mathfrak{gl}(V) \oplus V,$$

where i is the inclusion map and define operations:

$$l_2 = \llbracket \cdot, \cdot \rrbracket, \quad l_3 = -(-1)^{|z||x|}T.$$

Theorem 4.2. *With notations above, the omni-Lie superalgebra $(\mathcal{E}, \llbracket \cdot, \cdot \rrbracket, \langle \cdot, \cdot \rangle)$ defines a Lie 2-superalgebra $(V \xrightarrow{d} \mathfrak{gl}(V) \oplus V, l_2, l_3)$.*

Proof. For Condition (a), by the grading in $\mathfrak{gl}(V) \oplus V$ we have $\deg(A + x) = \deg(A) = \deg(x)$, then

$$\begin{aligned}
&\llbracket A + x, B + y \rrbracket + (-1)^{|x||y|}\llbracket B + y, A + x \rrbracket \\
&= [A, B] + \frac{1}{2}(Ay - (-1)^{|x||y|}Bx) + (-1)^{|A||B|}[B, A] \\
&\quad + (-1)^{|x||y|}\frac{1}{2}(Bx - (-1)^{|y||x|}Ay) \\
&= [A, B] + (-1)^{|x||y|}[B, A] + \frac{1}{2}(Ay - (-1)^{|x||y|}Bx) \\
&\quad + \frac{1}{2}((-1)^{|x||y|}Bx - Ay) \\
&= 0.
\end{aligned}$$

Conditions (b), (c), (e) and (f) are easy to be checked. By Proposition 3.2, we have $l_3 = J_2$, thus Conditions (g)–(h) hold. For Condition (i), we first verify a special case by taking $e_1 = A, e_2 = B, e_3 = C, e_4 = w$. In fact, by the definition

of l_3 and Proposition 3.2, we get

$$\begin{aligned}
& [A, l_3(B, C, w)] - (-1)^{|x||y|}[B, l_3(A, C, w)] \\
& + (-1)^{(|x|+|y|)|z|}[C, l_3(A, B, w)] - [l_3(A, B, C), w] \\
& - l_3([A, B], C, w) + (-1)^{|y||z|}l_3([A, C], B, w) - (-1)^{(|y|+|z|)|w|}l_3([A, w], B, C) \\
& - l_3(A, [B, C], w) + (-1)^{|z||w|}l_3(A, [B, w], C) - l_3(A, B, [C, w]) \\
= & -\frac{1}{8}A[B, C]w + \frac{1}{8}(-1)^{|x||y|}B[A, C]w - \frac{1}{8}(-1)^{(|x|+|y|)|z|}C[A, B]w + 0 \\
& + \frac{1}{4}[[A, B], C]w - \frac{1}{4}(-1)^{|y||z|}[[A, C], B]w + \frac{1}{4}(-1)^{|x||y|}(-1)^{|x||z|}[[B, C], A]w \\
& + \frac{1}{8}(-1)^{|x|(|y|+|z|)}[B, C]Aw - \frac{1}{8}(-1)^{|y||z|}[A, C]Bw + \frac{1}{8}[A, B]Cw \\
= & \frac{1}{4}\{[[A, B], C] + (-1)^{|x|(|y|+|z|)}[[B, C], A] + (-1)^{(|x|+|y|)|z|}[[C, A], B]\}w \\
& - \frac{1}{8}\{A[B, C] - (-1)^{|x||y|}B[A, C] + (-1)^{(|x|+|y|)|z|}C[A, B] \\
& - (-1)^{|x|(|y|+|z|)}[B, C]A + (-1)^{|y||z|}[A, C]B - [A, B]C\}w \\
= & \frac{1}{4}\{[[A, B], C] - [A, [B, C]] + (-1)^{|x||y|}[B, [A, C]]\}w \\
& - \frac{1}{8}\{[A, [B, C]] - (-1)^{|x||y|}[B, [A, C]] - [[A, B], C]\}w \\
= & 0.
\end{aligned}$$

The general case can be checked similarly. \square

A Lie 2-superalgebra is called strict if $l_3 = 0$. This kind of Lie 2-superalgebras can be described in terms of crossed module.

Definition 4.3. A crossed module of Lie superalgebras consists of a pair of Lie superalgebras $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ together with an action of \mathfrak{g} on \mathfrak{h} and a homomorphism $\varphi : \mathfrak{h} \rightarrow \mathfrak{g}$ such that

$$\varphi(xh) = [x, \varphi(h)]_{\mathfrak{g}}, \quad \varphi(h)k = [h, k]_{\mathfrak{h}}, \quad \forall h, k \in \mathfrak{h}, \forall x \in \mathfrak{g}.$$

Proposition 4.4. Strict Lie 2-superalgebras are in one-to-one correspondence with crossed modules of Lie superalgebras.

Proof. Let $\mathcal{V}^1 \xrightarrow{d} \mathcal{V}^0$ be a strict Lie 2-superalgebra. Define $\mathfrak{g} = \mathcal{V}^0$, $\mathfrak{h} = \mathcal{V}^1$, and the following two brackets on \mathfrak{g} and \mathfrak{h} :

$$\begin{aligned}
[h, k]_{\mathfrak{h}} &= l_2(dh, k) = [dh, k], \quad \forall h, k \in \mathfrak{h} = \mathcal{V}^1; \\
[x, y]_{\mathfrak{g}} &= l_2(x, y) = [x, y], \quad \forall x, y \in \mathfrak{g} = \mathcal{V}^0.
\end{aligned}$$

Obviously, $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is a Lie superalgebra by (a) and (g) in Definition 4.1. By Condition (h), we have

$$\begin{aligned}
& -[[h, k]_{\mathfrak{h}}, l]_{\mathfrak{h}} + (-1)^{|k||l|}[[h, l]_{\mathfrak{h}}k]_{\mathfrak{h}} + [h, [k, l]_{\mathfrak{h}}]_{\mathfrak{h}} \\
= & -[d[dh, k], l] + (-1)^{|k||l|}[d[dh, l], k] + [dh, [dk, l]] \\
= & -[[dh, dk], l] + (-1)^{|k||l|}[[dh, dl], k] + [dh, [dk, l]] \\
= & 0.
\end{aligned}$$

This means that $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ is also Lie superalgebra. By Condition (e) and taking $\varphi = d$, we have

$$\varphi([h, k]_{\mathfrak{h}}) = d([dh, k]) = [dh, dk] = [\varphi(h), \varphi(k)]_{\mathfrak{g}},$$

which implies that φ is a homomorphism of Lie superalgebras. Next we define an action of \mathfrak{g} on \mathfrak{h} by

$$xh \triangleq l_2(x, h) = [x, h] \in \mathfrak{h},$$

which is an action because the equality,

$$\begin{aligned} & [x, y]h - x(yh) + (-1)^{|x||y|}y(xh) \\ &= [[x, y], h] - [x, [y, h]] + (-1)^{|x||y|}[y, [x, h]] \\ &= 0, \end{aligned}$$

holds by Condition (h). Finally, it is easy to check that

$$\begin{aligned} \varphi(xh) &= d([x, h]) = [x, dh] = [x, \varphi(h)]_{\mathfrak{g}} \\ \varphi(h)k &= [dh, k] = [h, k]_{\mathfrak{h}}. \end{aligned}$$

Therefore, we obtain a crossed module of Lie superalgebras.

Conversely, a crossed module of Lie superalgebras gives rise to a Lie 2-superalgebra with $d = \varphi$, $\mathcal{V}^0 = \mathfrak{g}$, $\mathcal{V}^1 = \mathfrak{h}$, $l_3 = 0$ and the following operations:

$$\begin{aligned} l_2(x, y) &\triangleq [x, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}; \\ l_2(x, h) &\triangleq xh, \quad \forall x \in \mathfrak{g}; \\ l_2(h, k) &\triangleq 0. \end{aligned}$$

All of the conditions for a Lie 2-superalgebra can be verified directly from the definition of a crossed module. \square

Another kind of Lie 2-superalgebras is called **skeletal** if $d = 0$. As pointed in [8], Skeletal Lie 2-superalgebras are in one-to-one correspondence with quadruples $(\mathfrak{g}, V, \rho, l_3)$ where \mathfrak{g} is a Lie superalgebra, V is a super vector space, ρ is a representation of \mathfrak{g} on V and l_3 is a 3-cocycle on \mathfrak{g} with values in V . See [14] for more details of the cohomology of Lie superalgebras.

Example 4.5. Given a quadratic Lie superalgebra $(\mathfrak{g}, [\cdot, \cdot], B)$ over \mathbb{K} , where B is the supertrace $B(x, y) = \text{str}(xy)$ by Kac [7], a skeletal Lie 2-superalgebra can be constructed as follows: $\mathcal{V}^1 = \mathbb{K}$, $\mathcal{V}^0 = \mathfrak{g}$, $d = 0$ and

$$l_2(x, y) = [x, y], \quad l_2(x, h) = 0, \quad l_3(x, y, z) = B([x, y], z) \quad (12)$$

where $x, y, z \in \mathfrak{g}, h \in \mathbb{K}$. Condition (i) is from the fact that Cartan 3-form l_3 is closed. That is, by the invariance of B , we have

$$\begin{aligned} & \delta l_3(w, x, y, z) \\ &= B(w, [x, [y, z]]) + B(w, [x, [y, z]]) + (-1)^{|y||z|}B(w, [[x, z], y]) \\ & \quad + (-1)^{(|x|+|y|)|z|}B(w, [z, [x, y]]) - B(w, [[x, y], z]) - (-1)^{|x||y|}B(w, [y, [x, z]]) \\ &= 2B(w, [x, [y, z]]) + (-1)^{|y||z|}[[x, z], y] - [[x, y], z]) \\ &= 0, \end{aligned}$$

which holds from the super Jacobi identity on \mathfrak{g} . Therefore, $(\mathbb{K} \xrightarrow{0} \mathfrak{g}, l_2, l_3)$ is a Lie 2-superalgebra as a super version of the string Lie algebra.

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